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Viscous flow through pipes of various cross-sections

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Abstract
An interesting variety of pipe cross-sectional shapes can be generated, for which the Navier–Stokes equations can be solved exactly. The simplest cases include the known solutions for elliptical and equilateral triangle cross-sections. Students can find pipe cross-sections from solutions of Laplace's equation in two dimensions, and then plot the velocity distribution in the pipe. The total flow for a given pressure gradient and pipe area can be readily compared for different pipe shapes.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Fluid mechanics is difficult because the Navier–Stokes equations describing viscous flow are nonlinear. Flow in a pipe of fixed cross-section is an exception: for steady incompressible flow the continuity equation and the Navier–Stokes force-balance equations reduce to [1–4]

$$\partial_x v_x + \partial_y v_y + \partial_z v_z = 0 \tag{1}$$

$$\begin{aligned}
\rho(v_x\partial_x + v_y\partial_y + v_z\partial_z)v_x &= -\partial_x p + \eta\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)v_x \\
\rho(v_x\partial_x + v_y\partial_y + v_z\partial_z)v_y &= -\partial_y p + \eta\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)v_y \\
\rho(v_x\partial_x + v_y\partial_y + v_z\partial_z)v_z &= -\partial_z p + \eta\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)v_z.
\end{aligned} \tag{2}$$

Here v_x is the x component of the velocity, ρ is the mass density, p the pressure and η the viscosity. Consider steady flow in a pipe of uniform cross-section, with its axis coincident with the z-axis. Then the transverse velocity components v_x and v_y are zero, so the transverse pressure gradients $\partial_x p$ and $\partial_y p$ are zero also (the pressure at given z is the same across the pipe). From fluid conservation, expressed in equation (1), $\partial_z v_z = 0$: the flow velocity is the same at given transverse coordinates x, y along the pipe, and we are left with

$$\left(\partial_x^2 + \partial_y^2\right)v_z = \frac{1}{\eta}\partial_z p. \tag{3}$$

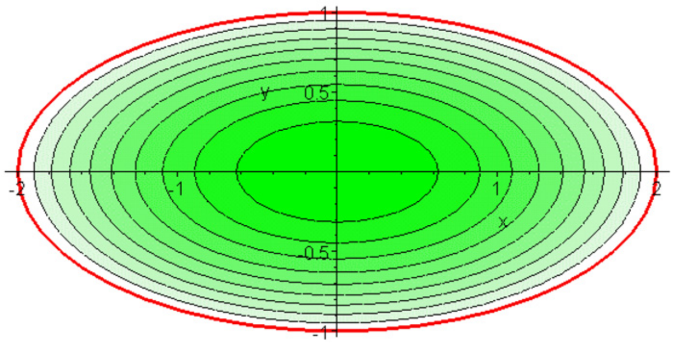


Figure 1. Velocity contours, at 10% to 90% of the maximum velocity, in an elliptical pipe with $a = 2b$.

Since v_z does not depend on the longitudinal coordinate z, the longitudinal pressure gradient $\partial_z p$ must be constant. A solution of (3) satisfying the boundary condition that $v_z(x, y)$ is to be zero at the pipe wall then provides the velocity profile, and the rate of total fluid flow Q can be found by integration.

A simple example, to be used for comparison later, is the elliptical cross-section [1–5] given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The velocity profile (here u is the velocity at the pipe centre)

$$v_z = u\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{4}$$

satisfies the boundary condition that v_z is zero on the pipe wall, and satisfies (3) provided

$$u = -\frac{1}{2\eta}\partial_z p\frac{a^2b^2}{a^2+b^2}. \tag{4a}$$

The rate of total flow through the pipe is

$$\begin{aligned} Q &= 4u\int_0^a \mathrm{d}x \int_0^{b\sqrt{1-x^2/a^2}} \mathrm{d}y\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \\ &= \frac{8}{3}ub\int_0^a \mathrm{d}x\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{3}} 2 = \frac{\pi}{2}uab. \end{aligned} \tag{5}$$

Hence

$$Q = \frac{\pi}{4\eta}(-\partial_z p)\frac{(ab)^3}{(a^2+b^2)} \tag{6}$$

and

$$v_z = \frac{2Q}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{7}$$

Figure 1 shows the velocity contours: all are ellipses with the same major-to-minor axis ratio.

In this paper we shall show how students can generate pipe shapes for which the velocity profile can be found exactly. The method is not new [2, 4, 5] but does not seem to have been applied beyond the solution for flow in a pipe of equilateral triangular cross-section. It is presented here as a pedagogical entry to viscous hydrodynamics, and for the interesting pipe shapes and flows that can be generated.

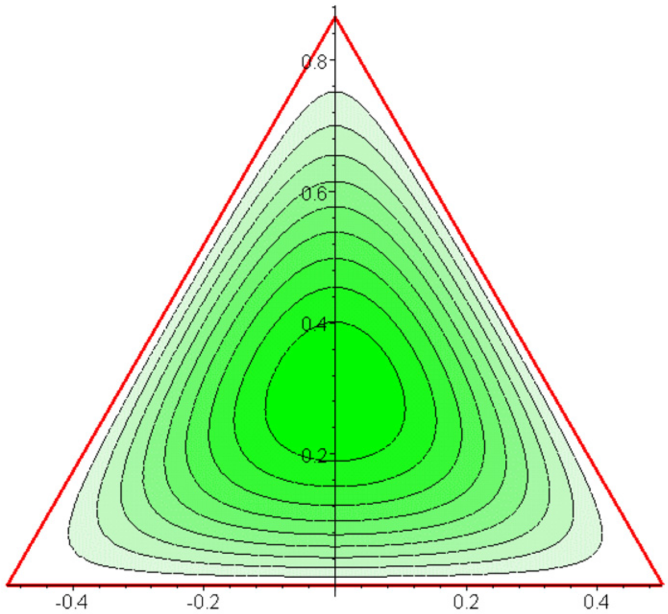


Figure 2. Velocity distribution in an equilateral triangle pipe (equation (8)). The contours are at 0.1 (0.1) 0.9 of the maximum velocity.

2. Functions of $x + iy$, and $(x + iy)^3$ in particular

The expression $\alpha x^2 + \beta y^2$ has Laplacian $2(\alpha + \beta)$. This is unchanged by addition of terms proportional to xy, x or y, or constants. Thus a solution of (3) can be obtained by setting v_z equal to any harmonic function $f(x, y)$ (one for which $\left(\partial_x^2 + \partial_y^2\right) f = 0$) plus $\alpha x^2 + \beta y^2$, plus terms bilinear or linear in x and y or constant, provided $\alpha + \beta = \frac{1}{2\eta}(\partial_z p)$. It is well known, and easily shown by differentiation, that both the real and imaginary parts of any twice-differentiable function of the variable $x + iy$ are harmonic.

Solution of (3) is only half the problem, however: one must also ensure that v_z is zero at the pipe wall. Let $W(x, y) = 0$ define the pipe wall or walls. For example, an equilateral triangle of side a with base resting on the x-axis ($y = 0$) has walls defined by

$$\begin{aligned} W_3 &= y\left(x + \frac{y}{\sqrt{3}} - \frac{a}{2}\right)\left(x - \frac{y}{\sqrt{3}} + \frac{a}{2}\right) \\ &= x^2 y - \frac{1}{3}y^3 + \frac{1}{\sqrt{3}}ay^2 - \frac{1}{4}a^2 y. \end{aligned} \tag{8}$$

We notice that $(x + iy)^3 = x^3 - 3xy^2 + i\left(3x^2 y - y^3\right)$ has an imaginary part which corresponds to the first two terms in the expanded part of (8) (the real part is the same function with x and y interchanged and of opposite sign). This has zero Laplacian; so setting $v_z = uW_3$ will be a solution of (3) and satisfy $v_z = 0$ on the walls, provided

$$u = \frac{\sqrt{3}}{2a}\frac{1}{\eta}\partial_z p. \tag{9}$$

Figure 2 shows contours of equal velocity for the equilateral triangular pipe cross-section. Note that, in contrast to the elliptical pipe, the contours of constant v_z are not scaled-down versions of the pipe walls, but change from circular near the centre to triangular at the walls.

The rate flow through the equilateral triangular pipe is

$$Q_3 = 2u \int_0^{a/2} \mathrm{d}x \int_0^{\sqrt{3}\left(\frac{a}{2}-x\right)} \mathrm{d}y\, W_3 = -\frac{ua^5}{160}$$
$$= \frac{\sqrt{3}}{320} a^4 \left(-\frac{1}{\eta}\partial_z p\right). \tag{10}$$

It is interesting to compare the flow efficacy of various shapes. For a circular pipe we have, from (6) with $b = a$,

$$Q_c = \frac{\pi}{8}\left(-\frac{1}{\eta}\partial_z p\right) a^4, \qquad \frac{Q_c}{(\text{area})^2} = \frac{1}{8\pi}\left(-\frac{1}{\eta}\partial_z p\right). \tag{11}$$

The area of an equilateral triangle of side a is $\frac{\sqrt{3}}{4}a^2$, so

$$\frac{Q_3}{(\text{area})^2} = \frac{1}{20\sqrt{3}}\left(-\frac{1}{\eta}\partial_z p\right). \tag{12}$$

The value of $Q/(\text{area})^2$ is smaller than that for the circular cross-section by the factor $\frac{8\pi}{20\sqrt{3}} \approx 0.7255$. The ratio for elliptic to circular pipes is $2ab/(a^2 + b^2) \leqslant 1$.

3. Pipe shapes derived from $(x + \mathrm{i}y)^4$

Since $(x + \mathrm{i}y)^3$ gives the solution for an equilateral triangle, one might guess that $(x + \mathrm{i}y)^4$ does the same for a square. Not so, but it can come close, as we shall see. We have

$$(x + \mathrm{i}y)^4 = x^4 - 6x^2y^2 + y^4 + 4\mathrm{i}xy(x^2 - y^2). \tag{13}$$

The $x \leftrightarrow y$ symmetry of the real part makes it easier to use; in accordance with the general scheme set out at the beginning of section 2, we put

$$v_z = ua^{-4}[x^4 - 6x^2y^2 + y^4 - 2\beta a^2(x^2 + y^2) + a^4] \tag{14}$$

where a is a scale length and β is a dimensionless number. Then $(\partial_x^2 + \partial_y^2)v_z = -8\beta u/a^2$, so

$$u = \frac{a^2}{8\beta}\left(-\frac{1}{\eta}\partial_z p\right). \tag{15}$$

The pipes encompassed in (14) are closed areas (if any) bounded by curves $v_z = 0$. The topology depends on the parameter β. Let $r = \sqrt{x^2 + y^2}$ be the distance from the z-axis, and ϕ the azimuthal angle. Since $(x + \mathrm{i}y)^4 = r^4 \mathrm{e}^{4\mathrm{i}\phi}$, we can write v_z as

$$v_z = ua^{-4}[r^4 \cos 4\phi - 2\beta a^2 r^2 + a^4]. \tag{16}$$

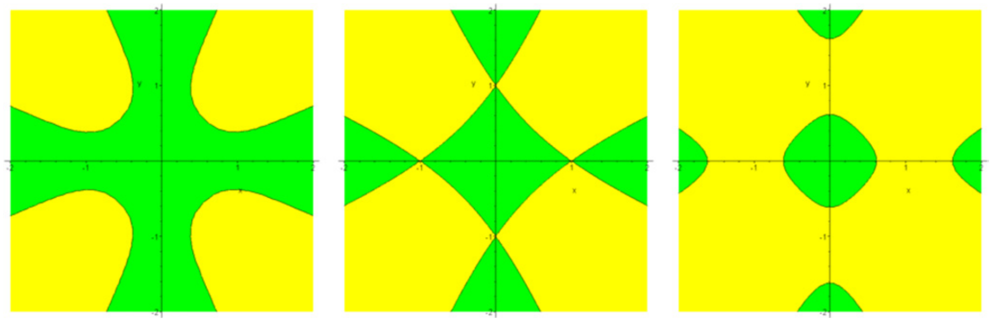
The pipe walls, if these exist, are given by $r_\pm(\phi)$, where

$$\frac{r_\pm^2}{a^2} = \frac{\beta \pm \sqrt{\beta^2 - \cos 4\phi}}{\cos 4\phi}. \tag{17}$$

For $\beta < 1$ there are four open branches in the xy plane, and no 'pipe' is enclosed (figure 3(a)). At $\beta = 1$ there is a cusped enclosed area, touching open branches when $\cos 4\phi = 1$, i.e., when $\phi = 0, \pi/2, \pi$ and $3\pi/2$. This is shown in figure 3(b). For $\beta > 1$ there is an enclosed area centred on the origin, and four open branches; an example is shown in figure 3(c).

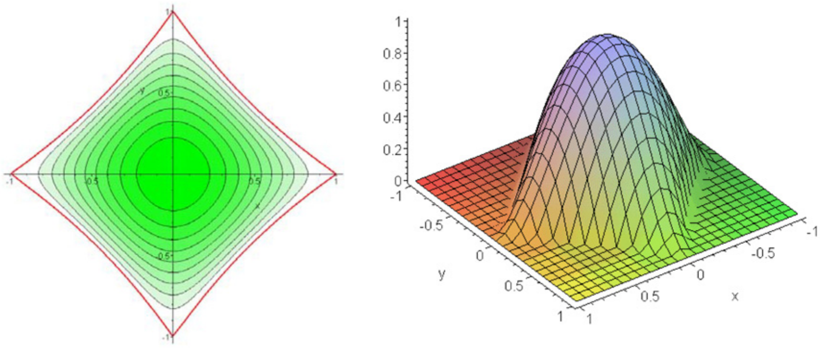
For $\beta \geqslant 1$ the pipe radius as a function of azimuthal angle is given by r_- in equation (17). The maximum radius is at $\cos 4\phi = 1$:

$$r_-^{\max} = a\left\{\beta - \sqrt{\beta^2 - 1}\right\}^{\frac{1}{2}}. \tag{18}$$



(a) (b) (c)

Figure 3. The three different topologies arising from equation (14) as β varies. For $\beta < 1$ there are open regions only, illustrated for $\beta = \frac{1}{2}$ in (a). At $\beta = 1$ the open areas pinch off a cusped enclosed area, shown in (b). As β increases from unity the enclosed area shrinks and becomes rounded, shown in (c) for $\beta = \frac{3}{2}$.



(a) (b)

Figure 4. Fluid velocity in the $(x + \mathrm{i}y)^4$ pipe, obtained from equation (16) with $\beta = 1$. Part (a) shows the contours at 0.1, 0.2, . . . , 0.9 of the maximum velocity, while (b) shows $v/v_{\max}$ in elevation.

The minimum radius is at $\cos 4\phi = -1$ ($\phi = \pi/4, 3\pi/4, 5\pi/4$ and $7\pi/4$),

$$r_-^{\min} = a\left\{\sqrt{\beta^2 + 1} - \beta\right\}^{\frac{1}{2}}. \tag{19}$$

For a square centred on the origin, $r^{\max}/r^{\min} = \sqrt{2}$, and this ratio of $r_-^{\max}$ to $r_-^{\min}$ is obtained for $\beta^2 = 25/24$.

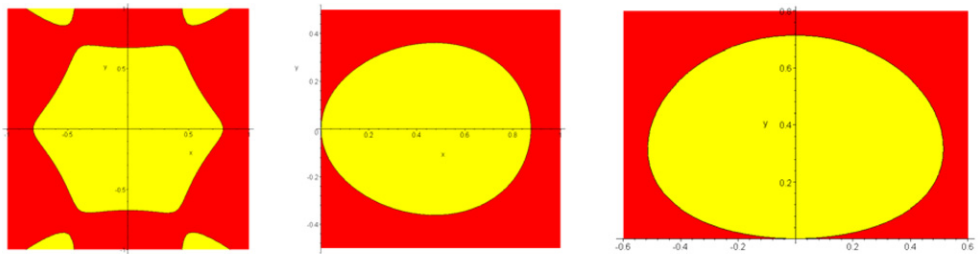
Figure 4 shows the velocity distribution in the $\beta = 1$ pipe. This pipe has cross-sectional area and rate of flow given by

$$\text{area} = a^2 2\ln(\sqrt{2} + 1) \approx 1.7627a^2 \tag{20}$$

$$Q_4 = ua^2\{2\ln(\sqrt{2} + 1) - 2\sqrt{2}/3\} \approx 0.8199\, ua^2. \tag{21}$$

Comparison with (11), with the use of (15), shows that the $Q/(\text{area})^2$ ratio is smaller than that for a circular pipe equation (11) by the factor

$$\frac{\pi\{2\ln(\sqrt{2} + 1) - 2\sqrt{2}/3\}}{[2\ln(\sqrt{2} + 1)]^2} \approx 0.8290. \tag{22}$$



(a) (b) (c)

Figure 5. Pipe profiles obtained by setting $v_z = 0$ in (a) equation (24) with $\beta = 1$; (b) equation (25) with $\alpha = 1$, $\beta = 2$; (c) equation (26) with $\alpha = 1 = \beta$. The scale length a is unity in all the figures.

For comparison, a square pipe [6–8] has $Q/(\text{area})^2$ related to that for the circular pipe by the larger factor

$$\frac{2\pi}{3}\left[1 - 6\sum_{n=0}^{\infty}\frac{\tanh[(2n+1)\pi/2]}{[(2n+1)\pi/2]^5}\right] \approx 0.8833. \tag{23}$$

As β increases from unity the enclosed area shrinks, becoming more circular. I have not found analytical formulae for the area or rate of flow for general β.

4. Other pipe shapes and discussion

There is an uncountable infinity of analytic functions, and thus of solvable pipe shapes. Three simple illustrations are $(x + \mathrm{i}y)^6$, $\sin(x + \mathrm{i}y)$ and $\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)}$. Introducing a scaling length a, and dimensionless parameters α and β, we can write down the following velocity profiles based on the real parts:

$$v_z = ua^{-6}\{r^6\cos 6\phi - 2\beta a^4 r^2 + a^6\} \tag{24}$$

$$v_z = u\left\{\sin\frac{x}{a}\cosh\frac{y}{a} - \alpha\frac{x^2}{a^2} - \beta\frac{y^2}{a^2}\right\} \tag{25}$$

$$v_z = u\left\{\mathrm{e}^{-y/a}\cos\frac{x}{a} + \alpha\frac{x^2}{a^2} + \beta\frac{y^2}{a^2} - 1\right\}. \tag{26}$$

The $(x+\mathrm{i}y)^6$ example produces enclosed pipes with hexagonal symmetry for $\beta \geqslant (27/32)^{\frac{1}{3}} = 3/2^{\frac{5}{3}} \approx 0.9449$. As β increases from this value the pipes shrink, and become more circular. Figure 5 shows three particular cases of the three profiles above.

Students could be encouraged to create their own functions of $x + \mathrm{i}y$ which, with the addition of $\alpha x^2 + \beta y^2$ and terms in xy, x, y or constants, give enclosed areas in the xy plane. Then they can have the satisfaction of having solved, exactly, a problem in laminar viscous flow, quite possibly for the first time. The exercise has no obvious practical application, but it does teach some of the elements of hydrodynamics, and gives students a feel for the velocity distribution in a pipe, and how it depends on the pipe cross-sectional shape. The class experience may benefit from the comparison of the results obtained by different students or groups of students, with flow efficiencies $Q/(\text{area})^2$ being calculated by analytic or numeric integration.

Acknowledgment

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